Exercise sheets for Open Quantum Optical Systems

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(Dated: Winter Semester 2017/2018)

These document contains the exercise sheets of the different chapters that we have gone through in the Open Quantum Optical Systems course at the Erlangen-Nürnberg Friedrich-Alexander Universität (goo.gl/o96XWP). The notation and tone matches exactly that of the lecture notes for the course (goo.gl/JAugr3). The sheet for each chapter is divided in two parts. First, I provide the key questions that I think students should know how to answer, which will form the backbone of the exam. The goal of these questions is to help students know what they should study. Second, I provide an exercise that can be worked out with the tools developed in the chapter, and which students can get extra points from.

II. QUANTIZATION OF THE ELECTROMAGNETIC FIELD AS A COLLECTION OF HARMONIC OSCILLATORS

A. Key questions of the chapter

1. Starting from the Maxwell equations, derive the wave equation for the vector potential. Explain the quasi-1D approximation and, assuming perfectly conducting boundary conditions, write the corresponding vector potential in terms of appropriate mode functions. Interpret physically the conditions that such boundaries impose on the expansion coefficients and the allowed wave vectors.

2. Using the wave equation, prove that the expansion coefficients satisfy the dynamical equations of independent harmonic oscillators. Show that choosing a proper normalization factor in the vector potential expansion, the electromagnetic energy turns into the Hamiltonian for the independent oscillators. Quantize the electromagnetic field.

3. Take the Hamiltonian of a harmonic oscillator in terms of its quadratures, $\hat{H}_\text{o} = \hbar\omega(\hat{X}^2 + \hat{P}^2)$ with $[\hat{X}, \hat{P}] = 2\text{i}$. Use the expansion of the quadratures in terms of annihilation and creation operators to diagonalize the Hamiltonian, proving that the oscillator's Hilbert space is infinite dimensional. Discuss the two main physical consequences that quantum physics imposes on the oscillator: energy quantization and absence of well-defined phase-space trajectories.

4. Introduce the Wigner function as the quasi-probability density function in phase space that has the right marginals for quadrature measurements. Explain why it cannot be interpreted as a true probability density function and argue that the statistics of quantum mechanics cannot be simulated with standard noise, and hence quantum physics goes beyond classical physics.

5. Introduce coherent states as eigenstates of the annihilation operator and prove that they can be written as a displaced vacuum. Find their representation in the Fock-state basis, and discuss the photon-number distribution they lead to.

6. Introduce Gaussian states and find a condition based on coherent states that any state $\hat{\rho}$ must satisfy to be Gaussian. Prove that coherent states are Gaussian and find their corresponding Wigner function. Interpret it from the point of view of high-precision measurements and introduce squeezed states.

7. Introduce the concept of maximally-mixed state in finite dimension. Explain how the concept is extended to infinite dimension and introduce thermal states of a general Hamiltonian. Apply the definitions to the harmonic oscillator Hamiltonian $\hat{H} = \hbar\omega\hat{a}^\dagger\hat{a}$, writing the corresponding thermal state in the Fock basis. Prove that the state is Gaussian and compare its Wigner function with that of vacuum.

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B. Graded exercise 1: Cat states

Consider the state $|\alpha_{\text{cat}}\rangle = N_\alpha(|\alpha\rangle + |-\alpha\rangle)$, where $|\pm\alpha\rangle$ are coherent states, we take $\alpha \in \mathbb{R}$, and N is a suitable normalization constant. This state is known as a *cat state* because it is a quantum superposition of two coherent states, which are the most classical quantum states in the sense discussed in the lectures.

1. *Find the factor N_α that normalizes the state.*
   Hint: just operate on $\langle\alpha_{\text{cat}}|\alpha_{\text{cat}}\rangle = 1$

2. *Find the Fock-state representation of the state and discuss the corresponding photon-number probability distribution.*
   Hint: just write the Fock representations of each coherent state and simplify.

3. *Find the quantum characteristic function $\langle\hat{D}(x,p)\rangle$ and show that it is not Gaussian.*
   Hint: Use the complex representation of the displacement operator defining $\beta = (x + \mathrm{i}p)/2$, together with the normal form $\hat{D}(\beta) = e^{-|\beta|^2/2}e^{\beta\hat{a}^\dagger}e^{-\beta^*\hat{a}}$. Use then the fact that coherent states are eigenstates of the annihilation and creation operators, and you should be able to write the characteristic function as the sum of four exponentials quadratic in x and p.

4. *Find the Wigner function. Set $x = 0$ and discuss the shape and negativities that you can observe along the p axis.*
   Hint: Just Fourier-transform the characteristic function with the help of the Gaussian integral $\int_{\mathbb{R}} dz e^{Bz - z^2/2A} = \sqrt{2\pi A}e^{AB^2/2}$ valid for any $A > 0$ and $B \in \mathbb{C}$.

5. *Find the probability density functions for measurements of the $\hat{X}$ and $\hat{P}$ quadratures. Compare them with the ones corresponding to the mixed state $\hat{\rho}_\alpha = N_\alpha^2\left[|\alpha\rangle\langle\alpha| + |-\alpha\rangle\langle-\alpha| + 2\exp(-2\alpha^2)|0\rangle\langle 0|\right]$.*
   Hint: Find the marginals of the Wigner function with the help of the Gaussian integral of the previous question. Note that the definition of the Wigner function is linear in the density operator, and hence, the Wigner function of the mixed state $\hat{\rho}_\alpha$ is just a mixture of the Wigner function of the states that form $\hat{\rho}_\alpha$.

III. QUANTUM THEORY OF ATOMS AND THE TWO-LEVEL APPROXIMATION

A. Key questions of the chapter

1. Explain the main difference between the energy spectrum of the harmonic oscillator and the energy spectrum of an atomic system (you can use the Hydrogen atom as an example).

2. Consider hydrogen-like atoms, that is, atoms with all electrons in (inactive) closed shells, except for a single valence electron. Define the parity operator as the unitary transformation which inverts the sign of the relative atomic coordinate. Show that it also inverts the sign of the relative momentum. Assuming that the atomic Hamiltonian is symmetric under parity transformations, show that the energy eigenstates with the same parity cannot be connected by the relative-coordinate operator.

3. Justify the two-level approximation on an atom when working with monochromatic light. Introduce the Pauli pseudo-spin operators associated to the two atomic states, and write the atomic Hamiltonian in terms of $\hat{\sigma}_z$ within this approximation.

4. Show that a general atomic state can be written as $\hat{\rho} = (\hat{I} + \mathbf{b}^T \hat{\boldsymbol{\sigma}})/2$, where $b_j = \langle \hat{\sigma}_j \rangle$ and $\mathbf{b}^2 \leq 1$. Argue that $\mathbf{b} = 0$ corresponds to the maximally-mixed state, while states are pure if and only if $\mathbf{b}^2 = 1$. Introduce the Bloch space, and identify the points where the eigenstates of the Pauli operators are located.

5. Write down a general time-dependent Hamiltonian, and write down the Bloch equations (both in ordinary and complex form).

6. Consider an atom evolving freely. Use the Bloch equations to introduce the concept of (pseudo-)spin precession, and its visualization in Bloch space.

7. Consider now the semi-classical Rabi Hamiltonian. Using the description of light-matter interaction within the dipole approximation introduced in the next chapter, argue that this corresponds to the interaction of the two-level atomic system with a classical monochromatic field. Write down the complex Bloch equations and introduce the rotating-wave approximation, justifying it by averaging the equations over an optical cycle. Particularize the solution provided in the notes to the case in which the atom starts in the ground state. Discuss the frequency and amplitude of the oscillations performed by the excited-state population. What happens when the detuning between the atomic transition and the light beam is very large? How does this justify the two-level approximation?

B. Graded exercise 2: Beyond the rotating-wave approximation.

Consider the semi-classical Rabi Hamiltonian $\hat{H}(t) = \hbar\varepsilon\hat{\sigma}_z/2 + \hbar\Omega\cos(\omega t)\hat{\sigma}_x$, acting on a two-level atom initially in the ground state.

1. *Writing the state as* $|\psi(t)\rangle = a(t)e^{\mathrm{i}\varepsilon t/2}|g\rangle + b(t)e^{-\mathrm{i}\varepsilon t/2}|e\rangle$ *with* $a(0) = 1$ *and* $b(0) = 0$*, use the Schrödinger equation to show that the amplitudes obey the evolution equations*

$$\dot{a} = -\mathrm{i}\Omega e^{-\mathrm{i}\varepsilon t}\cos(\omega t)b, \tag{1a}$$

$$\dot{b} = -\mathrm{i}\Omega e^{\mathrm{i}\varepsilon t}\cos(\omega t)a. \tag{1b}$$

2. *Within the rotating-wave approximation, neglect fast-oscillating terms in this equations (or use the rotating-wave Hamiltonian* $\hat{H}(t) = \hbar\varepsilon\hat{\sigma}_z/2 + \hbar\Omega\left(e^{\mathrm{i}\omega t}\hat{\sigma} + e^{-\mathrm{i}\omega t}\hat{\sigma}^\dagger\right)/2$ *to find the equations of motion), and show that they lead to the solutions*

$$b(t) = -\mathrm{i}\frac{\Omega}{\sqrt{\Delta^2+\Omega^2}}e^{-\mathrm{i}\Delta t/2}\sin\left(\frac{\sqrt{\Delta^2+\Omega^2}}{2}t\right), \tag{2a}$$

$$a(t) = \cos\left(\frac{\sqrt{\Delta^2+\Omega^2}}{2}t\right) - \mathrm{i}\frac{\Delta}{\sqrt{\Delta^2+\Omega^2}}\sin\left(\frac{\sqrt{\Delta^2+\Omega^2}}{2}t\right), \tag{2b}$$

with $\Delta = \omega - \varepsilon$*.*
Hint: Starting from the equations for a and b, you can find a second order differential equation for b with constant coefficients, which you can easily solve by, e.g., an exponential ansatz.

3. *Next you are going to apply a perturbative approach on the original equations* (1)*. Under the assumption that* Ω *is small, expand the amplitudes as the power series*

$$a(t) = \sum_{n=0}^{\infty}\Omega^n a^{(n)}(t), \quad \textit{and} \quad b(t) = \sum_{n=0}^{\infty}\Omega^n b^{(n)}(t). \tag{3}$$

Plug this expansion into the evolution equations and match the terms of the same power in Ω *to show that they lead to the recurrent equations*

$$\dot{a}^{(n)} = -\mathrm{i}\cos(\omega t)e^{-\mathrm{i}\varepsilon t}b^{(n-1)}(t), \qquad [n = 1, 2, 3, ...] \tag{4a}$$

$$\dot{b}^{(n)} = -\mathrm{i}\cos(\omega t)e^{\mathrm{i}\varepsilon t}a^{(n-1)}(t), \tag{4b}$$

starting at $a^{(0)}(t) = 1$ *and* $b^{(0)}(t) = 0$*.*

4. *Working on resonance,* $\omega = \varepsilon$*, prove that to second order in* Ω *we get*

$$b(t) = -\frac{\mathrm{i}}{2}\Omega t + \frac{\Omega}{4\varepsilon}\left(1 - e^{2\mathrm{i}\varepsilon t}\right). \tag{5a}$$

$$a(t) = 1 - \frac{\Omega^2 t^2}{8} + \left(\frac{\Omega}{4\varepsilon}\right)^2\left[3 - (1 + 2\mathrm{i}\varepsilon t)\,e^{-2\mathrm{i}\varepsilon t} - 2\cos(2\varepsilon t)\right]. \tag{5b}$$

5. *Compare this solution with the rotating-wave solutions* (2) *for times* $t \ll \Omega^{-1}$*. Use this comparison to argue that effects beyond the rotating-wave approximation are suppressed by powers of* Ω/ε*.*

IV. LIGHT-MATTER INTERACTION

A. Key questions of the chapter

1. Introduce the description of light-matter interaction based on the dipole approximation for matter.

2. Apply this description to the interaction between the electromagnetic field of a cavity and a single atom to derive the Rabi and Jaynes-Cummings Hamiltonians, explaining the conditions under which these are expected to hold.

3. Introduce the dressed states as the eigenstates of the Jaynes-Cummings Hamiltonian. Discuss the corresponding energy spectrum, and compare it with that of the non-interacting case. Introduce also the concept of avoided crossing from the analysis of the first two energy transitions.

4. Find the evolution of an initial state consisting of an atom in the ground state and the field in an arbitrary pure state. Evaluate the excited-state population, and introduce the concept of quantum Rabi oscillations by assuming that the field starts in a Fock state.

5. Discuss the evolution of the excited-state population for a field starting in a coherent state. Make a qualitative plot showing the relevant physical phenomena: semiclassical Rabi oscillations, collapses, and revivals. Use approximations and reasonable arguments to find the time scale for each of these phenomena.

6. Introduce the dipole model of a dielectric. Use it to show how Maxwell equations inside of them are modified with respect to the ones in vacuum.

7. Discuss the effect that the linear term of the polarization density has on the field entering the dielectric: wavelength and amplitude reduction.

8. Use the macroscopic Maxwell equations to derive a wave equation for the electric field where the nonlinear terms of the polarization density act as a forcing or source term. Discuss then the frequency conversion phenomena linked to second order nonlinearity: second-harmonic generation, sum-frequency generation, difference-frequency generation, and down-conversion.

9. Write down the general interaction Hamiltonian of the electromagnetic field with the nonlinear dielectric. Assume that only two cavity modes with frequencies ω_0 and $\omega_2 \approx 2\omega_0$ and orthogonal polarization are relevant. Use energy and momentum conservation arguments to write down the down-conversion Hamiltonian.

10. Perform a classical (parametric) approximation for the pump mode, and find the Hamiltonian of the down-converted mode within this approximation. Explain qualitatively the types of behavior that this Hamiltonian leads to, particularly the difference between the stable and unstable regimes.

B. Graded exercise 3: Third-order (Kerr) nonlinearity

Consider an optical cavity containing a dielectric medium without inversion symmetry. $\mathbf{P}^{(2)}(\mathbf{r},t)$ vanishes identically fur such media, for which $\mathbf{P}^{(3)}(\mathbf{r},t)$ becomes then the dominant nonlinear term of the polarization density.

1. *Neglecting all cavity modes but one with frequency ω polarized along the x direction, show that*

$$\hat{P}_j^{(3)}(\mathbf{r}) = -\mathrm{i}\frac{\chi^{(3)}_{jxxx}}{\varepsilon_0^{1/2}}\left(\frac{\hbar\omega}{n_x LS}\right)^{3/2}\sin^3(n_x kz)(\hat{a}-\hat{a}^\dagger)^3. \tag{6}$$

   Hint: Just take the form of the single-mode field that we introduced in the lectures inside a dielectric medium, and apply the definition of the third-order nonlinear polarization density.

2. *Assume that the dielectric medium has length l and is placed right at the surface of the left mirror, hence covering the space $z \in [0,l]$ inside the cavity (as usual, we assume that the transverse dimensions of the dielectric medium cover the whole cavity). Use the light-matter interaction Hamiltonian within the dipole approximation, together with the rotating-wave approximation and the physical condition $n_x kl \gg 1$, to show that the dynamics of the optical mode is ruled by the Hamiltonian*

$$\hat{H} = \hbar(\omega+g)\hat{n} + \hbar g\hat{n}^2, \tag{7}$$

   with

$$g = \frac{9\chi^{(3)}_{xxxx} l\hbar\omega^2}{4\varepsilon_0 S n_x^2 L^2}. \tag{8}$$

   Hint: Just apply the definition of the light-matter Hamiltonian within the dipole approximation. Write the result of the spatial integral along z in terms of sinc functions that fall to zero when their argument is much larger than one. Then, after keeping the (free-)energy-conserving terms within the rotating-wave approximation, you'll need to use the canonical commutation relations to write the Hamiltonian in terms of the number operator.

3. *What are the eigenstates and eigenenergies of this Hamiltonian? Make a graphic representation of the energy levels for a given $|g| \ll \omega$, considering both the positive and negative cases, comparing it with the noninteracting situation $g = 0$.*

4. *Move to a picture rotating at the frequency $\omega+g$, so that the Hamiltonian in the new picture reads as $\tilde{H} = \hbar g\hat{n}^2$. Consider a coherent state $|\alpha\rangle$ as the initial state $|\psi(0)\rangle$. Find the state $\widetilde{|\psi(t)\rangle}$ in the rotating picture at subsequent times. Show that the photon-number distribution doesn't change in time. Show that the state is periodic, with period $T = 2\pi/g$ (take $g > 0$ from now on). Finally, show that at times $t_k = \pi(2k+1)/2g$ with $k = 0, 2, 4, \ldots$, the state is equivalent to the cat superposition*

$$\widetilde{|\psi(t_k)\rangle} = \frac{e^{-\mathrm{i}\pi/4}}{\sqrt{2}}\left(|\alpha\rangle + \mathrm{i}|-\alpha\rangle\right). \tag{9}$$

   Hint: Work in the Fock state basis.

V. QUANTUM OPTICS IN OPEN SYSTEMS

A. Key questions of the chapter

1. Introduce a model for the field outside a cavity with a partially-transmitting mirror (open cavity). For this, consider an external cavity that shares the partially-transmitting mirror with the main cavity. Write down the vector potential associated to the external cavity, and show that the infinite-length limit leads to a field composed of a continuous set of harmonic oscillators.

2. Focusing on a single mode of the main cavity, introduce the Hamiltonian describing its interaction with the external field. For this, argue that the beam-splitter Hamiltonian models the interaction term (photon tunneling), with a coupling parameter that can be taken independent of the frequency provided the mirror's reflectivity is large and slowly-varying with the frequency. Explain as well why we can introduce fictitious negative-frequency external modes that simplify enormously the math.

3. Working in the Heisenberg picture, introduce the equations modeling the dynamics of the cavity mode (quantum Langevin equation), showing that a damping term appears, together with an input operator that contains information about the state of the external field. Show that the presence of a monochromatic coherent component in the external field (laser injection) can be accounted for with a time-dependent Hamiltonian linear in the intracavity annihilation and creation operators. Evaluate also the type of statistics (expectation values and two-time correlators) that the input operator has when the external field is in a thermal state (argue that you can assume the same thermal state for all external modes approximately, that is, a frequency-independent thermal excitation number).

4. Working in the Schrödinger picture, derive an approximate evolution equation for the cavity mode. Start with the von Neumann equation for the density operator of the whole system (including the laser injection in the cavity Hamiltonian), and make two picture changes: first to a picture where the Hamiltonian becomes time-independent, and then to a picture where the free terms are removed, leaving only the (transformed) interaction term. Introduce the Born and non-backaction approximations, and show that they lead to an evolution equation that describes the dynamics of the cavity mode up to second order in the interaction. Trace out the external field using a thermal state as its initial state, finding the final Lindblad form for the master equation that rules the dynamics of the cavity mode.

5. Argue that the state of an empty cavity mode is Gaussian at all times (provided it starts in a Gaussian state). Find the first and second moments both through the quantum Langevin equations and through the master equation. Show that they reach an asymptotic state (independent of the initial condition), and interpret the corresponding state as a displaced thermal state.

6. Argue that a similar kind of procedure allows finding a master equation for an atom in free space. Write down such equation, derive the corresponding Bloch equations of the atom, and use them to introduce the concept of spontaneous emission. Show that starting in the excited state, the atomic density operator evolves through all the ground-excited mixed states until it reaches the ground state.

7. Starting from the Hamiltonian describing the interaction between the atom and the field in free space, argue (at least qualitatively) that the mixed states of the atom appear as a consequence of entanglement with the field.

8. Explain (also qualitatively) the effect that a frequency-dependent coupling between the atom and the field has on the master equation (Lamb shift).

B. Graded exercise 4: Atomic dephasing

While spontaneous emission is probably the most interesting incoherent process occurring in atoms from a theoretical point of view, there is another incoherent process which is even more important in practical terms, the so-called *dephasing*. Many different physical processes contribute to it, and in this exercise we will study it with a generic model: the atomic transition is affected by small random fluctuations (coming, for example, from stray random or thermal magnetic fields in the laboratory).

1. *Consider the Hamiltonian* $\hat{H} = \hbar[\varepsilon + \varphi(t)]\hat{\sigma}_z/2$, *where* $\varphi(t)$ *is a random variable with zero mean, that accounts for random fluctuations in the atomic transition frequency. Use the complex Bloch equations to show that the state, averaged over stochastic realizations, can be written as*
$$\hat{\rho}(t) = \frac{1}{2}\left[\hat{I} + b_z(0)\hat{\sigma}_z + 2e^{\mathrm{i}\varepsilon t - \Gamma^*(t)}b^*(0)\hat{\sigma} + 2e^{-\mathrm{i}\varepsilon t - \Gamma(t)}b(0)\hat{\sigma}^\dagger\right], \tag{10}$$
where we have defined
$$e^{-\Gamma(t)} = \overline{e^{-\mathrm{i}\int_0^t dt'\varphi(t')}}, \tag{11}$$
with $\Gamma(t) \in \mathbb{C}$ *in general and where the overbar denotes stochastic average.*

2. *Define* $\Phi(t) = \int_0^t dt'\varphi(t')$. *Provide arguments in favor of the identity* $\overline{\cos\Phi(t)}^2 + \overline{\sin\Phi(t)}^2 \leq 1$, *and use it to prove that* $\mathrm{Re}\{\Gamma(t)\} \geq 0$. *Assuming* $e^{-\mathrm{Re}\{\Gamma(t)\}}$ *decays to zero monotonically as a function of time, explain the trajectory followed in Bloch space by the initial superposition state* $|\psi(0)\rangle = \sqrt{p}|g\rangle + \sqrt{1-p}|e\rangle$ *with* $p \in [0,1]$. *Show that the final state is the incoherent mixture*
$$\lim_{t\to\infty} \hat{\rho}(t) = p|g\rangle\langle g| + (1-p)|e\rangle\langle e|, \tag{12}$$
so that the populations are left untouched, but the coherence has been destroyed.
Hint: Just evaluate the Bloch vector components at $t = 0$ using $|\psi(0)\rangle$, plug them in (10), and infer the Bloch vector components at any other time from it. In order to visualize the trajectory, it is useful to first forget about the z-precession at frequency ε; the remaining trajectory should just be a straight line (which becomes a spiral after including the precession).

3. *Show that the state* (10) *obeys the master equation*
$$\frac{d\hat{\rho}}{dt} = \left[-\frac{\mathrm{i}}{2}\left(\varepsilon + \mathrm{Im}\left\{\dot{\Gamma}(t)\right\}\right)\hat{\sigma}_z, \hat{\rho}\right] + \frac{\mathrm{Re}\left\{\dot{\Gamma}(t)\right\}}{2}\left(\hat{\sigma}_z\hat{\rho}\hat{\sigma}_z - \hat{\rho}\right), \tag{13}$$
where the overdot denotes time derivative.
Hint: Just evaluate the left and right hand sides independently, and show that they match.

4. *Consider the case in which* $\varphi(t)$ *derives from a Gaussian stochastic process. Defining* $\Phi(t) = \int_0^t dt'\varphi(t')$, *use the Gaussian-moments formula*
$$\overline{\Phi^n(t)} = \begin{cases} 0 & n \in \text{odd} \\ (n-1)!!\overline{\Phi^2(t)}^{n/2} & n \in \text{even} \end{cases}, \tag{14}$$
to show that
$$\overline{e^{-\mathrm{i}\Phi(t)}} = e^{-\overline{\Phi^2(t)}/2}. \tag{15}$$
Hint: Simply expand the exponential on the left hand side in Taylor series, use the formula, and simplify the expression to turn the elements of the sum into the ones required to obtain the right-hand-side's exponential.

5. *Consider the white-noise limit, that is,* $\overline{\varphi(t)\varphi(t')} = \gamma_\varphi\delta(t-t')$, *where* γ_φ *is some constant rate. Show that* $\Gamma(t) = \gamma_\varphi t/2$ *in this case, leading to a master equation*
$$\frac{d\hat{\rho}}{dt} = \left[-\frac{\mathrm{i}}{2}\varepsilon\hat{\sigma}_z, \hat{\rho}\right] + \frac{\gamma_\varphi}{4}\left(\hat{\sigma}_z\hat{\rho}\hat{\sigma}_z - \hat{\rho}\right), \tag{16}$$
which is the usual master equation written in quantum optical problems to account for dephasing.
Hint: Just use the definition of $\Gamma(t)$ and $\Phi(t)$, together with (15).

VI. ANALYZING THE EMISSION OF OPEN SYSTEMS

A. Key questions of the chapter

1. Working in the Heisenberg picture, and using reasonable approximations such as setting slowly-varying functions of the frequency to their value at the cavity resonance or ignoring retardation effects, write down the field coming out of an open cavity as a function of the intracavity and input annihilation operators (input-output relations).

2. Using the model of the previous chapter for the interaction between the cavity mode and the external modes, derive a formal solution for the external annihilation operators in terms of their final conditions at some future time and the cavity annihilation operator. Use this solution to find an alternative form of the output field's annihilation operator as a Fourier transformed of the external field's initial condition. Use it to show that the output annihilation and creation operators satisfy canonical commutation relations in time. Prove that the structure of the forward and backward quantum Langevin equations make the open-cavity model satisfy causality.

3. Introduce the qualitative model of a photodetector, explaining how a single photon is capable of creating a measurable electronic pulse. Explain that the combination of all the pulses generated by different photons create a macroscopic stochastic current, and write down the relation between its moments and the quantum moments of the number operator associated to the output field coming from a source. Inspired by this relation, define the correlation functions of the output field, and use the input-output relations to show that, at zero temperature, they are directly proportional to correlation functions of the source.

4. Introduce, without proving but explaining them carefully, the quantum regression theorem and the quantum regression formula for two-time correlators. Using the example of a radiating atom, provide a physical interpretation for the two-time correlation function associated to direct photodetection (coincidence correlation function) as the probability of getting two consecutive photodetection events separated by a given time interval.

5. Taking the master equation of the driven cavity as the starting point (in a picture rotating at the laser frequency), evaluate the (normalized) coincidence correlation function for two situations: zero temperature ($\bar{n} = 0$ but $\mathcal{E} \neq 0$) and zero drive ($\bar{n} \neq 0$ but $\mathcal{E} = 0$). You may use the properties of coherent states for the first one, and the quantum regression formula for the second. Interpret the results as photons arriving to the detector at random in the first case, and photons arriving in bunches in the second case.

6. Consider an atom subject to resonant driving and spontaneous emission (at zero temperature). Using the master equation in a picture rotating at the laser frequency, write down the Bloch equations. Turn them into a second-order differential equation for the excited-state population and find its value in the asymptotic, stationary limit. Show that it is never possible to obtain population inversion (that is, more population in the excited state than in the ground state), reaching a limit of 1/2 for infinite driving. Find a expression of the (normalized) coincidence correlation function as a function of the excited-state population and solve its equation of motion to find the full time-dependence of the correlation function. Use the result to introduce the concept of antibunching.

7. Introduce homodyne detection for the field coming out of a source, and show that the first and second moments of the corresponding photocurrent are proportional to normally-ordered moments of the quadratures. Introduce also the quadrature noise spectrum and the definition of squeezing for the fields coming out of an open source.

8. Write down the quantum Langevin equations of an open cavity containing a second-order nonlinear medium that provides down-conversion for the cavity mode (use the parametric approximation). Assuming zero detuning for the down-conversion process and working below threshold (damping rate larger than the down-conversion rate), write down decoupled equations for two orthogonal quadratures and solve them. Argue that the state is Gaussian, and study the asymptotic state by evaluating the mean vector and covariance matrix. How much squeezing can you get inside the cavity? Is the state a minimum-uncertainty one? How about the state of the output field? Evaluate the noise spectrum of the independent quadratures and answer this question.

B. Graded exercise 5: Coincidence correlation function of the below-threshold OPO

We have studied in class the squeezing properties of the light coming out of an optical parametric oscillator (OPO) below threshold and at resonance. Here we will now evaluate the coincidence correlation function, which is essential to understand the photon statistics of the field emitted by any system.

1. *Can you guess, based on physical arguments, whether the the photons emitted by the OPO will arrive at the photodetector bunched, antibunched, or at random?*

2. *Let us remind the solution for the slowly-varying position and momentum quadratures,*

$$\lim_{t\to\infty}\tilde{X}^{\varphi}(t)=\sqrt{2\gamma}\lim_{t\to\infty}\int_0^t d\tau' e^{-\lambda_\varphi\tau'}\tilde{X}^{\varphi}_{\mathrm{in}}(t-\tau'),\qquad\left[\varphi=0,\frac{\pi}{2}\right],\tag{17}$$

with $\lambda_0=\gamma-g$, $\lambda_{\pi/2}=\gamma+g$, $g\in[0,\gamma[$, *and where the input operators satisfy the statistical properties* $\langle\tilde{X}^{\phi}_{in}(t)\rangle=0$ *and* $\langle\tilde{X}^{\phi}_{in}(t)\tilde{X}^{\phi'}_{in}(t')\rangle=e^{\mathrm{i}(\phi'-\phi)}\delta(t-t')$. *Use it to show that*

$$\lim_{t\to\infty}\langle\tilde{X}^{\varphi}(t)\tilde{X}^{\varphi'}(t+\tau)\rangle=\frac{2\gamma}{\lambda_\varphi+\lambda_{\varphi'}}e^{\mathrm{i}(\varphi'-\varphi)}\begin{cases}e^{-\lambda_{\varphi'}\tau} & \text{for } \tau>0\\ e^{\lambda_\varphi\tau} & \text{for } \tau<0\end{cases}.\tag{18}$$

Hint: In class we evaluated $\lim_{t\to\infty}\langle\tilde{X}^{\varphi}(t)\tilde{X}^{\varphi}(t+\tau)\rangle$. Just follow similar steps.

3. *In class we saw the formula*

$$\langle\delta\hat{L}_1\delta\hat{L}_2\delta\hat{L}_3\delta\hat{L}_4\rangle=\langle\delta\hat{L}_1\delta\hat{L}_2\rangle\langle\delta\hat{L}_3\delta\hat{L}_4\rangle+\langle\delta\hat{L}_1\delta\hat{L}_3\rangle\langle\delta\hat{L}_2\delta\hat{L}_4\rangle+\langle\delta\hat{L}_1\delta\hat{L}_4\rangle\langle\delta\hat{L}_2\delta\hat{L}_3\rangle,\tag{19}$$

valid for Gaussian states when the operators $\hat{L}_j$ *are all linear in position and momentum, and* $\delta\hat{L}_j=\hat{L}_j-\langle\hat{L}_j\rangle$. *Argue that for the OPO below threshold (and any other problem described by linear quantum Langevin equations), this formula applies even when considering Heisenberg-picture operators at different times, that is, for the set* $\{\hat{L}_1(t_1),\hat{L}_2(t_2),\hat{L}_3(t_3),\hat{L}_4(t_4)\}$.

4. *Use the previous formula to show that the coincidence correlation function of the OPO below threshold can be written as*

$$\bar{G}^{(2)}(\tau)=\lim_{t\to\infty}\left[\left|\langle\hat{a}^\dagger(t)\hat{a}^\dagger(t+\tau)\rangle\right|^2+\left|\langle\hat{a}^\dagger(t)\hat{a}(t+\tau)\rangle\right|^2+\langle\hat{a}^\dagger(t)\hat{a}(t)\rangle^2\right].\tag{20}$$

Use then (18) *to obtain*

$$\lim_{t\to\infty}\langle\hat{a}^\dagger(t)\hat{a}^\dagger(t+\tau)\rangle=\frac{1}{4}\left[\frac{\sigma}{1-\sigma}e^{-(1-\sigma)\gamma\tau}+\frac{\sigma}{1+\sigma}e^{-(1+\sigma)\gamma\tau}\right],\tag{21a}$$

$$\lim_{t\to\infty}\langle\hat{a}^\dagger(t)\hat{a}(t+\tau)\rangle=\frac{1}{4}\left[\frac{\sigma}{1-\sigma}e^{-(1-\sigma)\gamma\tau}-\frac{\sigma}{1+\sigma}e^{-(1+\sigma)\gamma\tau}\right],\tag{21b}$$

where $\sigma=g/\gamma\in[0,1[$. *Show from these expressions that* $\bar{G}^{(2)}(\tau)$ *is a monotonically decreasing function of time. Does this match your answer in the first question of the exercise?*

Hint: For the first step, just apply the identity $\langle\hat{A}\hat{B}\rangle=\langle\hat{B}^\dagger\hat{A}^\dagger\rangle^*$. For the second part, just write the annihilation and creation operators as a function of the position and momentum quadratures, and use (18).

VII. EFFECTIVE MODELS: ELIMINATION OF SPURIOUS DEGREES OF FREEDOM

A. Key questions of the chapter

1. What is an effective model? Can you explain in which situations we might expect them to appear?

2. Consider a closed system described by a time-independent Hamiltonian. Define projector operators that divide the Hilbert space into relevant and irrelevant sectors, and use them to find an equation of motion for the state projected on the relevant sector. Define formally an effective Hamiltonian from these equations, showing that in general we obtain a non-Hermitian and time-dependent expression for it.

3. While for some problems the effective Hamiltonian becomes Hermitian under suitable conditions, for other it does not. For the latter, how do you interpret the fact that we cannot find a Hermitian effective Hamiltonian?

4. Show that the original Hamiltonian of the system can be split into a term that doesn't connect the relevant and irrelevant subspaces (*free Hamiltonian*), and another that does (*interaction Hamiltonian*). Find a simple expression for the effective Hamiltonian up to second order in the interaction. Use it to show that a far-detuned monochromatic optical beam can be used to generate a motional potential on the center of mass motion of an atom.

5. Consider now an open system described by a master equation with time-independent Liouvillian. Define projector superoperators that divide the space of operators into relevant and irrelevant sectors, and use them to find an effective (time-nonlocal) master equation for the relevant part of the state. Show that the original Liouvillian can be split into a term that does not connect the relevant and irrelevant subspaces (free Liouvillian), and another one that does (interaction Liouvillian). Approximate the effective master equation to second order in the latter.

6. Consider a bipartite Hilbert space structure, consisting in the tensor product of two degrees of freedom that we call system and environment. Consider physical situations in which, from the point of view of the system, the environment remains approximately frozen in its free-Liouvillian equilibrium state. Define a projector superoperator adapted to such situation. Particularize the previous effective master equation to this choice of projector, and show how it can be approximated by a time-local effective master equation provided that the correlation functions of the environment decay fast enough compared to all incoherent rates of the system's evolution.

7. Consider an atom at high temperature emitting in all directions. Consider as well one mode of an open cavity at zero temperature. Argue qualitatively how the atom can be effectively cool down and its emission directed by coupling it to the cavity via a Jaynes-Cummings interaction, provided that the coupling is strong (i.e., larger than the atomic spontaneous emission rate) but still much smaller than the cavity decay rate. Use the effective master equation derived above (taking the atom as the system and the cavity as the environment) to make this statement mathematically rigorous.

B. Graded exercises

Graded exercise 6: Some important effective Hamiltonians

1. ***Kerr from down-conversion.*** *Consider the down-conversion Hamiltonian*

$$\hat{H} = \hbar\omega_0\hat{a}^\dagger\hat{a} + \hbar\omega_2\hat{b}^\dagger\hat{b} + \mathrm{i}\hbar\frac{g_0}{2}(\hat{b}\hat{a}^{\dagger 2} - \hat{b}^\dagger\hat{a}^2), \tag{22}$$

in the large-detuning limit, $|\Delta| \gg g_0$, with $\Delta = \omega_0 - \omega_2/2$. Find an effective Hamiltonian for the down-converted mode, under the assumption that the pump mode is not populated initially. In particular, argue that $\hat{P} = |0\rangle_b\langle 0|$ is the appropriate projector in this case, with $\hat{b}|0\rangle_b = 0$, and show that the effective Hamiltonian has the Kerr form

$$\hat{H}_{\text{eff}} = \hbar\,(\omega - g)\,\hat{n} + \hbar g\hat{n}^2, \tag{23}$$

with $\hat{n} = \hat{a}^\dagger\hat{a}$ and $g = -g_0^2/8\Delta$.

2. ***Raman driving.*** *Consider the following problem: we would like to couple two atomic energy eigenstates whose transition does not couple to a laser (e.g., because they have the same parity or their transition frequency is not in the optical domain, but in the microwave domain, as usually happens with hyperfine states). You are going to show that there is an effective way of connecting them by using an additional atomic state and two far-detuned laser beams that couple it to the original states. Consider specifically the situation depicted in Fig. 1, described by the Hamiltonian*

$$\hat{H}(t) = -\hbar\varepsilon_g|g\rangle\langle g| - \hbar\varepsilon_e|e\rangle\langle e| + \hbar\left(\Omega_g e^{-\mathrm{i}\omega_g t}|a\rangle\langle g| + \Omega_g^* e^{\mathrm{i}\omega_g t}|g\rangle\langle a| + \Omega_e e^{-\mathrm{i}\omega_e t}|a\rangle\langle e| + \Omega_e^* e^{\mathrm{i}\omega_e t}|e\rangle\langle a|\right), \tag{24}$$

where we have taken the energy origin in the auxiliary level $|a\rangle$, and chosen the signs so that $\varepsilon_j > 0$.
Show that in a new picture defined by the transformation $\hat{U}_\text{c}(t) = e^{\hat{H}_\text{c}t/\mathrm{i}\hbar}$ with $\hat{H}_\text{c} = -\hbar\omega_g|g\rangle\langle g| - \hbar\omega_e|e\rangle\langle e|$, the Hamiltonian reads

$$\tilde{H} = \hbar\Delta_g|g\rangle\langle g| + \hbar\Delta_e|e\rangle\langle e| + \hbar\left(\Omega_g|a\rangle\langle g| + \Omega_g^*|g\rangle\langle a| + \Omega_e|a\rangle\langle e| + \Omega_e^*|e\rangle\langle a|\right), \tag{25}$$

with $\Delta_j = \omega_j - \varepsilon_j$.
In the far-detuned limit $|\Delta_j| \gg |\Omega_j|$, argue that starting from an unpopulated auxiliary level, the relevant sector of the Hilbert space is defined by the projector $\hat{P} = |g\rangle\langle g| + |e\rangle\langle e|$, leading to the effective Hamiltonian

$$\tilde{H}_\text{eff} \approx \hbar\Delta_g|g\rangle\langle g| + \hbar\Delta_e|e\rangle\langle e| + \hbar\Omega_\text{eff}|e\rangle\langle g| + \hbar\Omega_\text{eff}^*|g\rangle\langle e|, \tag{26}$$

with $\Omega_\text{eff} = \Omega_g\Omega_e^*/\Delta_g$.
Come back to the original picture and show that the Hamiltonian can be written in the common form (you'll need to make an energy shift)

$$\hat{H}_\text{eff} = \frac{\hbar\varepsilon}{2}\hat{\sigma}_z + \hbar\left(\Omega_\text{eff}e^{-\mathrm{i}\omega_\text{eff}t}\hat{\sigma}^\dagger + \Omega_\text{eff}^* e^{\mathrm{i}\omega_\text{eff}t}\hat{\sigma}\right), \tag{27}$$

where $\omega_\text{eff} = \omega_e - \omega_g$, $\varepsilon = \varepsilon_g - \varepsilon_e$, and the Pauli operators are defined in the usual way.
Hint: Just follow the steps we have developed in class, and be careful that at the end you will need to justify the approximation $\Omega_g\Omega_e^*/\Delta_g \approx \Omega_g\Omega_e^*/\Delta_e$ to get a Hermitian Hamiltonian.

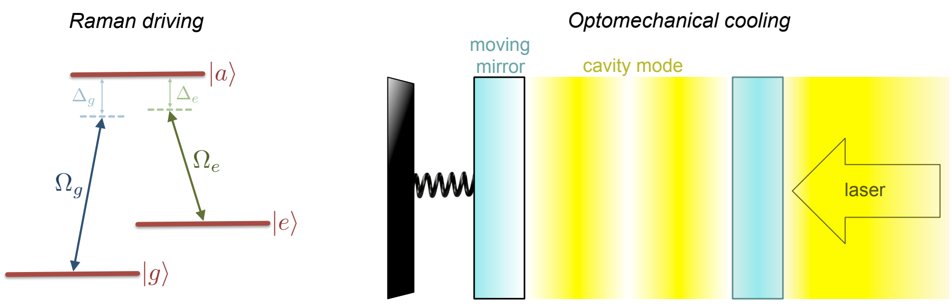

Figure 1. Sketches of the Raman driving and optomechanical cooling schemes.

Graded exercise 7: Optomechanical sideband cooling

In the last exercise of the course we will study one research-level problem: how to cool down a mechanical resonator by using laser light. It is research level in the sense that the theory behind it, which is what you we will do in this exercise, was developed only about a decade ago, and the related experiments started only around the same time. Altogether, these developments gave birth to the now well-established field of quantum optomechanics, which is currently a subject of active investigation by many groups all around the world.

1. *Consider an optical cavity with an the end mirror coupled to a fixed wall through a spring (Fig. 1), so that it can oscillate at some frequency* Ω *(much smaller than optical frequencies). Take the position of the mirror around its equilibrium location as a dynamical variable* $\hat{z}$*, whose evolution (in the absence of coupling to other systems) is provided by a harmonic oscillator Hamiltonian* $\hat{H}_\mathrm{m} = \frac{1}{2m}\hat{p}_z^2 + \frac{m\Omega^2}{2}\hat{z}^2$*, where* m *is the mirror's mass and* $\hat{p}_z$ *its momentum, so that* $[\hat{z}, \hat{p}_z] = \mathrm{i}\hbar$*. Consider now one mode of the cavity with annihilation operator* $\hat{a}$ *and resonance frequency* ω_c *when the cavity has length* L*. Using the fact that the cavity's length is* $L + \hat{z}$*, and assuming that the mirror's deviations from equilibrium are small compared to* L*, show that the Hamiltonian of the system can be approximated by*
$$\hat{H} = \hbar\Omega\hat{b}^\dagger\hat{b} + \hbar\omega_\mathrm{c}\hat{a}^\dagger\hat{a} + \hbar g_0\hat{a}^\dagger\hat{a}(\hat{b}^\dagger + \hat{b}), \tag{28}$$
where we have introduced the annihilation and creation operators for the mechanical oscillator, $\hat{b}$ *and* $\hat{b}^\dagger$*, respectively, and the so-called optomechanical coupling rate is given*
$$g_0 = -\omega_\mathrm{c}\frac{z_\mathrm{zpf}}{L}, \tag{29}$$
where $z_\mathrm{zpf} = \sqrt{\hbar/2\Omega m}$ *refers to the zero-point fluctuations of the mirror's position.*
Hint: You will need to use the expression that we saw at the beginning of the lectures for the resonance frequencies inside a cavity as a function of its length.

2. *Assume now that the temperature is low enough for the number of thermal photons at optical frequencies to be negligible, but large enough for the mechanical oscillator to have a lot of thermal excitations or phonons* $\bar{n}$ *(this is what naturally happens at room temperature, since typical optical frequencies are hundreds of THz while mechanical oscillations are never larger than GHz). Moreover, assume as well that the cavity is driven by a laser, so that the master equation of the whole system reads*
$$\frac{d\hat{\rho}}{dt} = \left[\frac{\hat{H}}{\mathrm{i}\hbar} + \left(\mathcal{E}e^{-\mathrm{i}\omega_\mathrm{L}t}\hat{a}^\dagger - \mathcal{E}^*e^{\mathrm{i}\omega_\mathrm{L}t}\hat{a}\right), \hat{\rho}\right] + \kappa\mathcal{D}_a[\hat{\rho}] + \gamma(\bar{n}+1)\mathcal{D}_b[\hat{\rho}] + \gamma\bar{n}\mathcal{D}_{b^\dagger}[\hat{\rho}]. \tag{30}$$
Show that there exists a transformation operator $\hat{U}_\mathrm{c}(t)$ *such that the state* $\hat{\rho}'(t) = \hat{U}_\mathrm{c}^\dagger(t)\hat{\rho}(t)\hat{U}_\mathrm{c}(t)$ *in the new picture satisfies the time-independent master equation*
$$\frac{d\hat{\rho}'}{dt} = \left[-\mathrm{i}\Omega\hat{b}^\dagger\hat{b} + \mathrm{i}\Delta_\mathrm{c}\hat{a}^\dagger\hat{a} + \mathcal{E}\hat{a}^\dagger - \mathcal{E}^*\hat{a} - \mathrm{i}g_0\hat{a}^\dagger\hat{a}(\hat{b}^\dagger + \hat{b}), \hat{\rho}'\right] + \kappa\mathcal{D}_a[\hat{\rho}'] + \gamma(\bar{n}+1)\mathcal{D}_b[\hat{\rho}'] + \gamma\bar{n}\mathcal{D}_{b^\dagger}[\hat{\rho}'], \tag{31}$$
where $\Delta_\mathrm{c} = \omega_\mathrm{L} - \omega_\mathrm{c}$.

3. *Assume that the state is coherent, that is,* $\hat{\rho}'(t) = |\beta(t)\rangle\langle\beta(t)|\otimes|\alpha(t)\rangle\langle\alpha(t)|$*, and show that the coherent amplitudes satisfy the equations of motion*
$$\dot{\alpha} = \mathcal{E} - \left[\kappa - \mathrm{i}\Delta_\mathrm{c} + \mathrm{i}g_0(\beta + \beta^*)\right]\alpha, \tag{32a}$$
$$\dot{\beta} = -(\gamma + \mathrm{i}\Omega)\beta - \mathrm{i}g_0|\alpha|^2. \tag{32b}$$
Hint: Find the equations of motion of $\langle\hat{a}\rangle$ and $\langle\hat{b}\rangle$ like we did in class for other problems starting from the master equation, and use the fact that coherent states are right and left eigenstates of the annihilation and creation operators, respectively.

4. *Consider now a displacement operator with time-dependent amplitude*
$$\hat{D}_c[\chi(t)] = e^{\chi(t)\hat{c}^\dagger - \chi^*(t)\hat{c}}, \tag{33}$$

where $\hat{c}$ and $\hat{c}^\dagger$ are annihilation and creation operators, respectively, that is, $[\hat{c},\hat{c}^\dagger]=1$. Show that its time-derivative can be written as

$$\frac{d\hat{D}_c[\chi(t)]}{dt}=\hat{D}_c[\chi(t)]\left(\dot{\chi}\hat{c}^\dagger-\dot{\chi}^*\hat{c}+\frac{\dot{\chi}\chi^*-\chi\dot{\chi}^*}{2}\right). \tag{34}$$

Hint: Take χ and χ^* as independent variables, so that you can write the time derivative as $\frac{d}{dt}=\dot{\chi}\frac{\partial}{\partial\chi}+\dot{\chi}^*\frac{\partial}{\partial\chi^*}$, and apply it to the normally-ordered form of the displacement operator $\hat{D}_c[\chi(t)]=e^{-|\chi(t)|^2/2}e^{\chi(t)\hat{c}^\dagger}e^{-\chi^*(t)\hat{c}}$.

5. *Move to another picture defined by the transformation operator $\hat{U}(t)=\hat{D}_b[\beta(t)]\hat{D}_a[\alpha(t)]$, where the amplitudes α and β satisfy the classical equations of motion (32). Show that the transformed state $\tilde{\rho}(t)=\hat{U}^\dagger(t)\hat{\rho}'\hat{U}(t)$ satisfies the master equation*

$$\frac{d\tilde{\rho}}{dt}=\left[-\mathrm{i}\Omega\hat{b}^\dagger\hat{b}+\mathrm{i}\Delta(t)\hat{a}^\dagger\hat{a}-\mathrm{i}[g(t)\hat{a}^\dagger+g^*(t)\hat{a}+g_0\hat{a}^\dagger\hat{a}](\hat{b}^\dagger+\hat{b}),\tilde{\rho}\right]+\kappa\mathcal{D}_a[\tilde{\rho}]+\gamma(\bar{n}+1)\mathcal{D}_b[\tilde{\rho}]+\gamma\bar{n}\mathcal{D}_{b^\dagger}[\tilde{\rho}], \tag{35}$$

where we have defined the dressed detuning $\Delta(t)=\Delta_\mathrm{c}-g_0\left[\beta(t)+\beta^*(t)\right]$ and the dressed optomechanical coupling $g(t)=g_0\alpha(t)$.
Hint: Make the time derivative of $\tilde{\rho}(t)$ and use (34), (31), and the displacement formulas $\hat{U}^\dagger(t)\hat{a}\hat{U}(t)=\hat{a}+\alpha(t)$ and $\hat{U}^\dagger(t)\hat{b}\hat{U}(t)=\hat{b}+\beta(t)$, to rewrite it as (35). You should obtain extra terms in the Hamiltonian linear in the annihilation and creation operators, but this terms vanish identically thanks to the classical equations (32).

6. *Next we are going to find an effective master equation for the mechanical oscillator. Hence, we make the choices (we assume from now on that the classical solution is time-independent)*

$$\mathcal{L}_\mathrm{S}[\tilde{\rho}]=\left[-\mathrm{i}\Omega\hat{b}^\dagger\hat{b},\tilde{\rho}\right]+\gamma(\bar{n}+1)\mathcal{D}_b[\tilde{\rho}]+\gamma\bar{n}\mathcal{D}_{b^\dagger}[\tilde{\rho}], \tag{36a}$$

$$\mathcal{L}_\mathrm{E}[\tilde{\rho}]=\left[\mathrm{i}\Delta\hat{a}^\dagger\hat{a},\tilde{\rho}\right]+\kappa\mathcal{D}_a[\tilde{\rho}], \tag{36b}$$

$$\mathcal{L}_1[\tilde{\rho}]=\left[-\mathrm{i}\underbrace{(g\hat{a}^\dagger+g^*\hat{a}+g_0\hat{a}^\dagger\hat{a})\hat{X}}_{\hat{H}_1/\hbar},\tilde{\rho}\right], \tag{36c}$$

where we have defined the mechanical position quadrature $\hat{X}=\hat{b}^\dagger+\hat{b}$. Show that the stationary state of the free cavity mode is vacuum, $\bar{\rho}_\mathrm{E}=|0\rangle\langle 0|$, and then prove that

$$\mathcal{P}\mathcal{L}_1\mathcal{P}[\hat{Y}]=0, \tag{37}$$

for any operator $\hat{Y}$, where the projector superoperator acts as $\mathcal{P}[\hat{Y}]=\mathrm{tr}_\mathrm{E}\{\hat{Y}\}\otimes\bar{\rho}_\mathrm{E}$.
Hint: Just apply the definition of the projector to (37), and you should be able to write it as something proportional to $\langle 0|\hat{a}|0\rangle$ and $\langle 0|\hat{a}^\dagger\hat{a}|0\rangle$.

7. *Note that the Hamiltonian can be written as $\hat{H}_1/\hbar=\sum_{m=1}^{3}g_m\hat{S}_m\otimes\hat{E}_m$, with the choices $g_2=g=g_1^*$, $g_3=g_0$, $\hat{S}_1=\hat{S}_2=\hat{S}_3=\hat{X}$, $\hat{E}_1=\hat{a}=\hat{E}_2^\dagger$, and $\hat{E}_3=\hat{a}^\dagger\hat{a}$. Show that the environmental operators are all closed on their own, and use the quantum regression formula to show that ($\tau>0$)*

$$\lim_{t\to\infty}\langle\hat{A}(t)\hat{a}(t+\tau)\hat{C}(t)\rangle_\mathrm{E}=\langle 0|\hat{A}\hat{a}\hat{C}|0\rangle_\mathrm{E}e^{-(\kappa-\mathrm{i}\Delta)\tau}, \tag{38a}$$

$$\lim_{t\to\infty}\langle\hat{A}(t)\hat{a}(t+\tau)\hat{C}(t)\rangle_\mathrm{E}=\langle 0|\hat{A}\hat{a}^\dagger\hat{C}|0\rangle_\mathrm{E}e^{-(\kappa+\mathrm{i}\Delta)\tau} \tag{38b}$$

$$\lim_{t\to\infty}\langle\hat{A}(t)\hat{a}^\dagger(t+\tau)\hat{a}(t+\tau)\hat{C}(t)\rangle_\mathrm{E}=\langle 0|\hat{A}\hat{a}^\dagger\hat{a}\hat{C}|0\rangle_\mathrm{E}e^{-2\kappa\tau}, \tag{38c}$$

for any environmental operators $\hat{A}$ and $\hat{C}$. Then, show that the cavity correlators required for the mechanical effective master equation read as

$$C_{mn}(\tau)=e^{-(\kappa+\mathrm{i}\Delta)\tau}\delta_{m1}\delta_{n2},\qquad K_{nm}(\tau)=e^{-(\kappa-\mathrm{i}\Delta)\tau}\delta_{n1}\delta_{m2}. \tag{39}$$

Hint: Follow the steps followed in the example of the lectures.

8. *Take $\hat{H}_\mathrm{S} = \hbar\Omega\hat{b}^\dagger\hat{b}$ as the only part of the mechanical evolution which is relevant within the decay of the cavity correlators. Show that*

$$\tilde{X}(\tau) = e^{\hat{H}_\mathrm{S}\tau/\mathrm{i}\hbar}\hat{X}e^{-\hat{H}_\mathrm{S}\tau/\mathrm{i}\hbar} = e^{\mathrm{i}\Omega\tau}\hat{b} + e^{-\mathrm{i}\Omega\tau}\hat{b}^\dagger, \tag{40}$$

and then prove that the effective mechanical master equation

$$\frac{d\tilde{\rho}_\mathrm{S}}{dt} = \mathcal{L}_\mathrm{S}[\tilde{\rho}_\mathrm{S}] + |g|^2\int_0^t d\tau\left[C_{12}(\tau)\hat{X}\tilde{\rho}_\mathrm{S}\hat{X}(\tau) - K_{12}(\tau)\hat{X}\hat{X}(\tau)\tilde{\rho}_\mathrm{S} + \mathrm{H.c.}\right], \tag{41}$$

can be written in the $t \gg \kappa^{-1}$ limit as

$$\frac{d\tilde{\rho}_\mathrm{S}}{dt} = \mathcal{L}_\mathrm{S}[\tilde{\rho}_\mathrm{S}] + \mathcal{L}_-[\tilde{\rho}_\mathrm{S}] + \mathcal{L}_+[\tilde{\rho}_\mathrm{S}] + \mathcal{L}_\mathrm{NRW}[\tilde{\rho}_\mathrm{S}], \tag{42}$$

with

$$\mathcal{L}_-[\tilde{\rho}_\mathrm{S}] = \Gamma_-^\mathrm{opt}\mathcal{D}_b[\tilde{\rho}] - \mathrm{i}[\delta\Omega_-\hat{b}^\dagger\hat{b},\tilde{\rho}], \quad \mathcal{L}_+[\tilde{\rho}_\mathrm{S}] = \Gamma_+^\mathrm{opt}\mathcal{D}_{b^\dagger}[\tilde{\rho}] - \mathrm{i}[\delta\Omega_+\hat{b}^\dagger\hat{b},\tilde{\rho}], \tag{43a}$$

$$\mathcal{L}_\mathrm{NRW}[\tilde{\rho}_\mathrm{S}] = |g|^2\left[\frac{\hat{b}\tilde{\rho}\hat{b}}{\kappa+\mathrm{i}(\Delta-\Omega)} + \frac{\hat{b}^\dagger\tilde{\rho}\hat{b}^\dagger}{\kappa+\mathrm{i}(\Delta+\Omega)} - \frac{\hat{b}^2\tilde{\rho}}{\kappa-\mathrm{i}(\Delta+\Omega)} - \frac{\tilde{\rho}\hat{b}^{\dagger 2}}{\kappa-\mathrm{i}(\Delta-\Omega)} + \mathrm{H.c.}\right], \tag{43b}$$

and

$$\Gamma_\mp^\mathrm{opt} = \frac{|g|^2/\kappa}{1+\left(\frac{\Delta\pm\Omega}{\kappa}\right)^2}, \quad \delta\Omega_\mp^\mathrm{opt} = \frac{|g|^2(\Delta\pm\Omega)}{\kappa^2+(\Delta\pm\Omega)^2}.$$

Hint: Just perform the time integrals (taking the limit $t/\kappa \gg 1$), and reorder the terms in the proper way.

9. *Argue that $\mathcal{L}_\mathrm{NRW}$ can be neglected within a rotating-wave approximation provided that*

$$\frac{|g|^2}{\sqrt{\kappa^2+(\Delta\pm\Omega)^2}} \ll 2\Omega. \tag{44}$$

Within this approximation, show that the effective mechanical master equation can be written as

$$\frac{d\tilde{\rho}_\mathrm{S}}{dt} = \left[-\mathrm{i}(\Omega+\delta\Omega_-+\delta\Omega_+)\hat{b}^\dagger\hat{b},\tilde{\rho}_\mathrm{S}\right] + \Gamma_-\mathcal{D}_b[\tilde{\rho}_\mathrm{S}] + \Gamma_+\mathcal{D}_{b^\dagger}[\tilde{\rho}_\mathrm{S}], \tag{45}$$

with

$$\Gamma_- = \gamma(\bar{n}+1) + \Gamma_-^\mathrm{opt}, \quad \Gamma_+ = \gamma\bar{n} + \Gamma_+^\mathrm{opt}, \tag{46}$$

which can be rewritten in the standard form

$$\frac{d\tilde{\rho}_\mathrm{S}}{dt} = \left[-\mathrm{i}\Omega_\mathrm{eff}\hat{b}^\dagger\hat{b},\tilde{\rho}_\mathrm{S}\right] + \Gamma_\mathrm{eff}(\bar{n}_\mathrm{eff}+1)\mathcal{D}_b[\tilde{\rho}_\mathrm{S}] + \Gamma_\mathrm{eff}\bar{n}_\mathrm{eff}\mathcal{D}_{b^\dagger}[\tilde{\rho}_\mathrm{S}], \tag{47}$$

with

$$\Omega_\mathrm{eff} = \Omega + \delta\Omega_- + \delta\Omega_+, \tag{48a}$$

$$\Gamma_\mathrm{eff} = \Gamma_- - \Gamma_+ = \gamma\left[1 + \frac{C}{1+\left(\frac{\Delta+\Omega}{\kappa}\right)^2} - \frac{C}{1+\left(\frac{\Delta-\Omega}{\kappa}\right)^2}\right], \tag{48b}$$

$$\bar{n}_\mathrm{eff} = \frac{\Gamma_+}{\Gamma_- - \Gamma_+} = \frac{\bar{n} + \frac{C}{1+\left(\frac{\Delta-\Omega}{\kappa}\right)^2}}{1 + \frac{C}{1+\left(\frac{\Delta+\Omega}{\kappa}\right)^2} - \frac{C}{1+\left(\frac{\Delta-\Omega}{\kappa}\right)^2}}, \tag{48c}$$

where we have defined the cooperativity $C = |g|^2/\gamma\kappa$.

10. *Inspect* $\bar{n}_{\text{eff}}$*, and argue that the best cooling is obtained when* $\Delta = -\Omega$ *(red sideband driving) and* $\Omega \gg \kappa$ *(resolved sideband regime). Under these conditions, show that when* $C \gg 1$*, then we have*

$$\Gamma_{\text{eff}} \approx \gamma C, \qquad \bar{n}_{\text{eff}} \approx \frac{\bar{n}}{C} + \frac{\kappa^2}{4\Omega^2}, \tag{49}$$

so that when $C \gg \bar{n}$*, we obtain the minimum effective thermal occupation* $\bar{n}_{\text{eff}} \approx \frac{\kappa^2}{4\Omega^2}$*.*

11. *Self-consistency check: show that all the approximations required to derive the effective master equation are compatible with the cooling conditions.*

12. *Finally, argue that the asymptotic state of this equation in the original Schrödinger picture corresponds to the displaced thermal state*

$$\lim_{t\to\infty} \hat{\rho}_{\text{S}}(t) = \hat{D}_b(\beta)\bar{\rho}_{\text{th}}(\bar{n}_{\text{eff}})\hat{D}_b^\dagger(\beta). \tag{50}$$